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Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal

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Abstract

This paper is concerned with a nonlocal evolution equation which is used to model the spatial dispersal of organisms. We study the existence, uniqueness and stability of the positive steady solution for this nonlocal evolution equation under general conditions. The global dynamics are also investigated and a trichotomy of the global asymptotics is established.

Keywords: Nonlocal evolution equation; Resolvent positive operator; Principal eigenvalue

1. Introduction

In this paper we study the existence, stability and uniqueness of steady states of the nonlinear evolution equation

$$\begin{cases} u_t(x,t) = \int_\Omega J(x,y)u(y,t)\,dy + b(x)u(x,t) + f(x,u(x,t)), \\ u(x,0) = \phi(x), \end{cases} \tag{1.1}$$

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where J, b and f are sufficiently smooth functions and J is positive. The mathematical model (1.1) was proposed by Hutson, Martinez, Mischaikow, and Vickers [14] (see also [13]) in their study of spatial dispersal of cells or organisms. Here the integral operator with kernel J is the dispersal operator (we also use J for the operator) and u is the density of a single species, which is considered in an n-dimensional habitat. The mechanism of dispersal is a major focus of theoretical interest and has received much attention recently. Most continuous models related to dispersal are based upon reaction–diffusion equations, which have been extensively studied (see [6,10,13,16,19]). In [14] the authors remark that the class of reaction–diffusion models, derived from random walks with branching process, is too restrictive to model the seed dispersal of a single species. They instead derive (1.1) from a variation of a position–jump process. Although the approach used to obtain (1.1) has similarities with the classical derivation of the Laplacian via random walk, it is not assumed that individuals move from a given patch with a binomial distribution. In contrast to the Laplacian, the integral operator in Eq. (1.1) is not a local operator, thus (1.1) should be considered as a model with long-range dispersal. For more details, we refer to [14] where Eq. (1.1) is carefully justified and shown to lead to complicated behavior. With the assumption that (1.1) has a positive steady solution $\tilde{u}$ and the nonlinearity $f(x,u) = u(a(x) - u)$, where $a(x)$ is C^1, in [14], it is shown that $\tilde{u}$ is the global attractor for all solutions whose initial data is non-trivial and nonnegative. As pointed out in [14], the mathematical analysis of (1.1) appears to be difficult even though the dispersal is represented by a bounded operator. Unlike the case of reaction–diffusion equations, the forward flow associated with (1.1) does not have a regularizing effect. Other nonlocal evolution equations with different nonlinearities have been derived and studied with basic theory being developed (see [3–5]) and we refer the reader to these for some background.

In the study of classical dispersal models, which are often based upon reaction–diffusion equations, many useful results on the global dynamics of diffusive equations were established in terms of principal eigenvalues of scalar elliptic eigenvalue problems. In [8], these linear elliptic eigenvalue problems are carefully explored. The authors obtained several important properties of the principal eigenvalue which were then used to study the global dynamics of logistic models. A trichotomy of the global asymptotics was also established. Meanwhile, by using monotone dynamical systems theory, in [9] the authors derived similar results for some quasimonotone reaction–diffusion systems with delays. Even though their approaches are not immediately applicable to (1.1) due to the lack of compactness, the importance of the principal eigenvalue is evident. Those approaches strongly suggest that an analogous idea using the principal eigenvalue should be developed, particularly for the case where the reaction term is sublinear (see [8] and (H4) below). In [14] the authors prove the existence of a principal eigenvalue for the integral operator $Lu := J * u + b(x)u$ under certain conditions, where $J * u = \int_\Omega J(x,y)u(y)\,dy$. We shall use those ideas combined with a comparison argument to study the steady solutions of (1.1) and their stability. Inspired by [2,8,18], our work focuses on the existence of a positive steady solution of (1.1) under more general assumptions than previously considered. We also study the uniqueness and stability of the solution and give a relatively complete description of the global dynamics of (1.1). The paper is organized as follows. In Section 2, we adopt the sub–supersolution methods to seek positive steady state solutions and to show uniqueness. In Section 3, we investigate the stability of this positive steady state and the long time behavior of solutions to (1.1).

2. Existence and uniqueness of positive steady solution

Let Ω be a bounded domain of $\mathbb{R}^n$ of class C^γ for some $\gamma \in (0,1)$. Let H be the Hilbert space $L^2(\Omega)$ with inner product $(\cdot,\cdot)$. Let $X := C(\overline{\Omega})$ be the Banach space of real continuous functions on $\overline{\Omega}$. Throughout this paper, X is considered as an ordered Banach space with a positive cone $X_+ = \{u \in X\colon\ u \geqslant 0\}$. It is well known that X_+ is generating, normal and has nonempty interior, which we denote by $\operatorname{int} X_+$ (see [1] for more details). For $\phi, \varphi \in X$, we write $\phi \leqslant \varphi$ if $\varphi - \phi \in X_+$, and $\varphi \gg \phi$ if $\varphi - \phi \in \operatorname{int} X_+$. An operator $T : X \to X$ is called positive if $TX_+ \subseteq X_+$.

Definition 2.1. An operator A is said to be resolvent positive if the resolvent set $\rho(A)$ of A contains an interval (α, ∞) and $(\lambda I - A)^{-1}$ is positive for sufficiently large $\lambda \in \rho(A) \cap \mathbb{R}$.

We also denote the spectral bound of an operator A by

$$\mathfrak{s}(A) = \sup\bigl\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(A)\bigr\}$$

where $\sigma(A)$ is the spectrum of A.

Consider the linear operator L on X defined by

$$Lu(x) := \int\limits_{\Omega} J(x, y)u(y)\,dy + b(x)u(x) \tag{2.1}$$

where we assume that

(H1) $J \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}^+)$ is symmetric, and
(H2) $J(x, y) > 0$, for any $x, y \in \overline{\Omega}$.

Lemma 2.2. *Let L be given by* (2.1). *Assume that* (H1) *and* (H2) *are satisfied and $b \in X$. Then L is a resolvent positive operator on $X$and $\mathfrak{s}(L) \in \sigma(L)$. If there exist $\Lambda \in \mathbb{R}$ and a continuous function $\phi \in X_+ \setminus \{0\}$ such that $L\phi = \Lambda\phi$, then $\Lambda = \mathfrak{s}(L)$. Moreover, $\mathfrak{s}(L)$ is an isolated eigenvalue and* $\ker(L - \mathfrak{s}(L)I) = \operatorname{span}\{\phi\}$.

Proof. First, we prove that L is a resolvent positive operator on X. In fact, L is a bounded, linear operator on X. Thus, $\rho(L)$ contains $(\|L\|, \infty)$. Choose $\omega \in \mathbb{R}^+$ such that $\omega > \sup_{x\in\overline{\Omega}} |\int_\Omega J(x, y)\,dy| + \sup_{x\in\overline{\Omega}} |b(x)|$. Obviously, $\lambda \in \rho(L)$ whenever $\lambda \geqslant \omega$. To prove that $(\lambda I - L)^{-1}$ is positive for all $\lambda \geqslant \omega$, it is sufficient to show that $(L - \lambda I)v \leqslant 0$ implies $v \geqslant 0$ for all $\lambda \geqslant \omega$. Let $v = v^+ - v^-$, where $v^- = \max\{-v, 0\}$ and $v^+ = \max\{v, 0\}$. If $v^- \neq 0$, then straight calculation, using $((L - \lambda)v, v^-) \leqslant 0$, yields that

$$0 \leqslant (J * v^+, v^-) \leqslant \bigl((L - \lambda)v^-, v^-\bigr)$$

and so

$$0 \leqslant \bigl((L - \lambda)v^-, v^-\bigr) \leqslant \left(\sup_{x\in\overline{\Omega}}\left|\int\limits_{\Omega} J(x, y)\,dy\right| + \sup_{x\in\overline{\Omega}} |b| - \lambda\right)(v^-, v^-) < 0.$$

The contradiction shows that $v^- \equiv 0$ in Ω, and so $v \geqslant 0$ in $\overline{\Omega}$. Thus, L is a resolvent positive operator.

Using Theorem 3.5 of [20], an extension of the Krein–Rutman theorem to resolvent positive operators, since $\sigma(L) \neq \emptyset$, we have $\mathfrak{s}(L) \in \sigma(L)$.

Now, assume that there exist $\phi \in X_+ \setminus \{0\}$ and $\Lambda \in \mathbb{R}$ such that

$$J * \phi = \bigl(\Lambda - b(x)\bigr)\phi.$$

It is easy to see that $J*\phi \gg 0$. Consequently, we have that $\phi \gg 0$ and $\Lambda - b(x) \gg 0$. If $\mathfrak{s}(L) > \Lambda$, then the linear operator K, defined by $Ku = (\mathfrak{s}(L) - b(x))u$, is continuous and bijective on X because $\mathfrak{s}(L) - b(x)$ is bounded and $\mathfrak{s}(L) - b(x) \gg 0$. Note that $(L - \mathfrak{s}(L)I)u = J*u - Ku$. We infer that the linear operator $L - \mathfrak{s}(L)I$ is Fredholm of index zero because $J : u \to J*u$ is compact on X (see [21, Theorem 5.C, p. 295]). By Propositions 2.3 and 2.4 [12, p. 151] (see also [15]), $\mathfrak{s}(L)$ is an eigenvalue with finite algebraic multiplicity and there exists a positive eigenfunction $\varphi \in X_+ \setminus \{0\}$ associated with $\mathfrak{s}(L)$. Since L is self-adjoint when considered as an operator on H, its eigenfunctions corresponding to distinct eigenvalues are orthogonal, hence we have $(\phi, \varphi) = 0$. But this contradicts the fact that both ϕ and φ are strictly positive. Thus, $\mathfrak{s}(L) = \Lambda$.

We now show that $\ker(L - \mathfrak{s}(L)) = \operatorname{span}\{\phi\}$. Suppose this is not true, then there exists an eigenfunction ψ associated with $\mathfrak{s}(L)$ such that $\psi \neq t\phi$ for all $t \in \mathbb{R}$. Since $\phi \gg 0$, there exist t such that $t\phi + \psi \geqslant 0$. Let $\bar{t} = \inf\{t \in \mathbb{R}\colon\ t\phi + \psi \geqslant 0\}$. Note that we have $J*(\bar{t}\phi + \psi) = (\mathfrak{s}(L) - b)(\bar{t}\phi + \psi)$ and $\bar{t}\phi + \psi \neq 0$. Again, $J*(\bar{t}\phi + \psi) \gg 0$ implies $\bar{t}\phi + \psi \gg 0$, which violates the definition of $\bar{t}$. The contradiction yields the desired conclusion.

To prove that $\mathfrak{s}(L)$ is isolated, we assume to the contrary that there exists a sequence $\{\mu_n\}_{n=1}^{\infty} \subset \sigma(L)$ such that $\lim_{n\to\infty} \mu_n = \mathfrak{s}(L)$ with $\mu_n \neq \mathfrak{s}(L)$ for all n. By Proposition 1 [21, p. 300], if n is sufficiently large, it is evident that $L - \mu_n I$ is a Fredholm operator of index zero and hence μ_n is an eigenvalue of L on X. On the other hand, thanks to $\mathfrak{s}(L) - b(x) \gg 0$, we have $\mu_n - b(x) \geqslant \delta$ for some $\delta > 0$, provided n is sufficiently large. Let θ_n be the corresponding eigenfunction with $\|\theta_n\|_X = 1$. Then $(\theta_n, \phi) = 0$. Due to the compactness of J and the fact that the sequence $\{\theta_n\}$ is bounded in X, along some subsequence, relabeled as $\{\theta_n\}$,

$$\lim_{n\to\infty} \big\| \big(\mu_n - b(x)\big)^{-1} J*\theta_n - \chi \big\|_X = 0$$

for some $\chi \in X$. From $(\mu_n - b(x))^{-1} J*\theta_n = \theta_n$, it follows that

$$\theta_n \to \chi, \quad \|\chi\|_X = 1, \quad \text{and} \quad \big(L - \mathfrak{s}(L)I\big)\chi = 0.$$

Thus, χ is an eigenfunction associated with $\mathfrak{s}(L)$. Since χ does not change sign in $\overline{\Omega}$ and is bounded away from zero, the convergence of θ_n to χ implies that $(\theta_n, \chi) > 0$ for large n, and we arrive at a contradiction again. Therefore, $\mathfrak{s}(L)$ is isolated and the proof is complete. □

Lemma 2.3. *Let all assumptions in Lemma* 2.2 *be satisfied. Then the following three statements are equivalent.*

(i) *There exists a* $\bar{u} \in X_+ \setminus \{0\}$ *such that* $-L\bar{u} \in X_+ \setminus \{0\}$.
(ii) $\mathfrak{s}(L) < 0$.
(iii) *For each* $f \in X$, $Lu = f$ *has exactly one solution in* X. *Moreover, if* w *is a solution to* $Lu = f$ *and* $f \leqslant 0$ *then* $w \geqslant 0$.

Proof. (i) ⇒ (ii). Suppose $\mathfrak{s}(L) \geqslant 0$. Let $g = L\bar{u} - \mathfrak{s}(L)\bar{u}$. Obviously, $g \leqslant 0$ and $g \neq 0$. Let ϕ be the positive eigenfunction associated with $\mathfrak{s}(L)$, then $(\phi, g) < 0$. On the other hand, we have

$$(\phi, g) = \big(\phi, \big(L - \mathfrak{s}(L)I\big)\bar{u}\big) = \big(\big(L - \mathfrak{s}(L)I\big)\phi, \bar{u}\big) = 0,$$

which is a contradiction. Thus, $\mathfrak{s}(L) < 0$.

(ii) ⇒ (i). This is trivial since the eigenfunction $\phi \gg 0$ and satisfies $L\phi \ll 0$.

(iii) ⇒ (ii). Clearly, (i) is true, thus $\mathfrak{s}(L) < 0$.

(ii) $\Rightarrow$ (iii). The existence of a unique solution is ensured by the fact that $0 \in \rho(L)$. Suppose w is the solution to $Lu = f$ and $f \leqslant 0$ with $w \notin X_+$. Let ϕ be the positive eigenfunction associated with $\mathfrak{s}(L)$. There exists $t > 0$ such that $t\phi + w \geqslant 0$. Once again, we let $\bar{t} = \inf\{t \in \mathbb{R}^+ \mid t\phi + w \geqslant 0\}$. Obviously, $\bar{t}\phi + w \neq 0$ and $L(\bar{t}\phi + w) \leqslant 0$. Now, let $x_0 \in \overline{\Omega}$ be a point such that $\bar{t}\phi(x_0) + w(x_0) = 0$, then we have

$$0 \leqslant \int_{\Omega} J(x_0, y)(\bar{t}\phi + w)\,dy = L(\bar{t}\phi + w)(x_0) \leqslant 0.$$

Since $J(x_0, y) > 0$ and $\bar{t}\phi + w$ is nonnegative, we have $\bar{t}\phi + w \equiv 0$. The contradiction leads to $\bar{t}\phi + w \gg 0$, which of course violates the definition of $\bar{t}$. Consequently, $w \geqslant 0$ and the proof is complete. □

Lemma 2.4. *Assume* (H1) *and* (H2). *Suppose that* $b_1, b_2 \in X$ *with* $b_1 \geqslant b_2$ *and* $b_1 \neq b_2$. *If* $L_i\phi_i = \mu_i\phi_i$, *where* $L_i u := J * u + b_i u$ *and* $\phi_i \in X_+ \setminus \{0\}$, *for* $i = 1, 2$, *then* $\mu_1 = \mathfrak{s}(L_1) > \mu_2 = \mathfrak{s}(L_2)$.

Proof. From Lemma 2.2, it follows that $\mu_1 = \mathfrak{s}(L_1)$ and $\mu_2 = \mathfrak{s}(L_2)$. Furthermore, $J * \phi_i = (\mu_i - b_i)\phi_i$, $i = 1, 2$ and $J * \phi_i \gg 0$ indicate $\phi_i \gg 0$, $i = 1, 2$. Now, if $\mu_1 \leqslant \mu_2$ then

$$(L_2 - \mu_2 I)\phi_1 = L_1\phi_1 + (b_2 - b_1)\phi_1 - \mu_2\phi_1 = (\mu_1 - \mu_2)\phi_1 + (b_2 - b_1)\phi_1 \leqslant 0.$$

According to Lemma 2.3, $\mathfrak{s}(L_2 - \mu_2) < 0$. This is impossible, because $\mathfrak{s}(L_2 - \mu_2) = 0$. Therefore, $\mathfrak{s}(L_1) > \mathfrak{s}(L_2)$. □

Next we consider the existence of solutions to

$$\int_{\Omega} J(x, y)u(y)\,dy + b(x)u(x) + f(x, u) = 0. \tag{2.2}$$

For the remainder of this paper we assume that

(H3) $f \in C^1(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ and $f(x, 0) \equiv 0$.
(H4) $f(x, \cdot)$ is strictly sublinear, i.e., for any $\alpha \in (0, 1)$, $f(x, \alpha s) > \alpha f(x, s)$, for all $s > 0$.

We also let

$$g(x, u) = \begin{cases} \frac{f(x,u)}{u} & \text{for } u > 0, \\ \partial_u f(x, 0) & \text{for } u = 0. \end{cases} \tag{2.3}$$

We will frequently use the facts that $f(x, u) = g(x, u)u$ and $g \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ is decreasing in $u \in \mathbb{R}^+$.

Definition 2.5. A function $u \in X$ is said to be a subsolution of (2.2) if

$$\int_{\Omega} J(x, y)u(y)\,dy + b(x)u(x) + f(x, u) \geqslant 0. \tag{2.4}$$

A supersolution is defined similarly by reversing the inequality.

Theorem 2.6. *Suppose that* (H1)–(H4) *are satisfied. If* (2.2) *has a supersolution* $\bar{u}$ *and subsolution* $\underline{u}$ *in* $X_+ \setminus \{0\}$ *such that* $\underline{u} \leqslant \bar{u}$, *then* (2.2) *has a unique nonzero solution in* X_+.

Proof. We define $U = \{u \in C(\overline{\Omega})\colon\ \underline{u} \leqslant u \leqslant \bar{u}\}$. By condition (H3), we see that $\frac{\partial f}{\partial u}(x,u)$ is uniformly bounded on $\overline{\Omega} \times U$ and

$$\frac{\partial f}{\partial u}(x,u) + \beta \gg 0 \tag{2.5}$$

for $(x,u) \in \overline{\Omega} \times U$ provided β is sufficiently large. For such β we define the mapping $T : U \to X$ by $v = Tu$ if

$$\int_{\Omega} J(x,y)v(y)\,dy + b(x)v(x) - \beta v(x) = -\big[f(x,u) + \beta u\big], \tag{2.6}$$

which is well defined since the linear operator

$$\mathcal{J}v(x) := \int_{\Omega} J(x,y)v(y)\,dy + b(x)v - \beta v(x)$$

is invertible on X. Next, we show that T is monotone in the sense that $w_1 \leqslant w_2$ implies $Tw_1 \leqslant Tw_2$, provided both w_1 and w_2 belong to U. In fact, if $w_1 \leqslant w_2$ then

$$Fw_1 \equiv f(x,w_1) + \beta w_1 \leqslant Fw_2 \equiv f(x,w_2) + \beta w_2,$$

thanks to (2.5). Notice that

$$\mathcal{J}(Tw_i) = -Fw_i,$$

thus, we have

$$\mathcal{J}(Tw_2 - Tw_1) \leqslant 0.$$

By virtue of Lemma 2.2, $(-\mathcal{J})^{-1}$ is a positive operator if β is sufficiently large. Hence, we obtain

$$Tw_1 \leqslant Tw_2.$$

From this, we deduce that the sequence defined inductively by

$$u_1 = T\bar{u} \quad \text{and} \quad u_n = Tu_{n-1}$$

is monotone decreasing. Similarly,

$$v_1 = T\underline{u} \quad \text{and} \quad v_n = Tv_{n-1}$$

defines a monotone increasing sequence. Furthermore, we can show by induction that

$$\underline{u} \leqslant v_1 \leqslant \cdots \leqslant v_n \cdots \leqslant u_n \leqslant \cdots \leqslant u_1 \leqslant \bar{u}.$$

Because the sequences $\{u_k\}$ and $\{v_k\}$ are monotone, the pointwise limits

$$u^* = \lim_{k\to\infty} u_k(x) \quad \text{and} \quad v^* = \lim_{k\to\infty} v_k(x)$$

both exist. Obviously, $\underline{u} \leqslant v^* \leqslant u^* \leqslant \bar{u}$. By the monotone convergence theorem, we have

$$\lim_{k\to\infty} \int_{\Omega} J(x,y)u_k(y)\,dy = \int_{\Omega} J(x,y)u^*(y)\,dy \quad \text{and}$$

$$\lim_{k\to\infty} \int_{\Omega} J(x,y)v_k(y)\,dy = \int_{\Omega} J(x,y)v^*(y)\,dy.$$

On the other hand, by the continuity of f, we see

$$\lim_{k\to\infty} f\big(x, u_k(x)\big) + b(x)u_{k+1}(x) = f(x, u^*) + b(x)u^*(x), \quad \text{and}$$
$$\lim_{k\to\infty} f\big(x, v_k(x)\big) + b(x)v_{k+1}(x) = f(x, v^*) + b(x)v^*(x).$$

Since

$$\int_\Omega J(x,y)u_k(y)\,dy = -\big[b(x)u_k(x) + f\big(x, u_{k-1}(x)\big)\big] + \beta\big[u_k(x) - u_{k-1}(x)\big],$$

it follows that

$$\int_\Omega J(x,y)u^*(y)\,dy + b(x)u^* + f(x,u^*) = 0.$$

Similarly,

$$\int_\Omega J(x,y)v^*(y)\,dy + b(x)v^* + f(x,v^*) = 0.$$

Next, we show that both u^* and v^* belong to X_+, i.e., (2.2) has at least one positive continuous solution. To this end, we make the following observations. First, we see that $J * v^* \in \operatorname{int} X_+$. This together with (H3) implies that both u^* and v^* are bounded away from zero. Second, due to the fact that $J * u^* + [b(x) + g(x,u^*)]u^* = 0$ and u^* is strictly positive and bounded on $\overline{\Omega}$, we may conclude that there is $\delta > 0$ such that

$$b(x) + g(x, u^*) \leqslant -\delta, \quad \text{for all } x \in \overline{\Omega}.$$

Furthermore, for any $x_1, x_2 \in \overline{\Omega}$, we find that

$$\begin{aligned} &J * u^*(x_1) - J * u^*(x_2) + \big[b(x_1) - b(x_2)\big]u^*(x_1) + \big[f\big(x_1, u^*(x_1)\big) - f\big(x_2, u^*(x_1)\big)\big] \\ &\quad = -\big[b(x_2) + \partial_u f\big(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)\big)\big]\big(u^*(x_1) - u^*(x_2)\big), \end{aligned} \tag{2.7}$$

where $0 \leqslant \theta \leqslant 1$. Without loss of generality, we may assume $u^*(x_1) \geqslant u^*(x_2)$. Since (H4) implies that $g(x,\cdot)$ is decreasing and $\partial_u f(x,s) \leqslant g(x,s)$ for all $s > 0$, the following inequalities are true:

$$\partial_u f\big(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)\big) \leqslant g\big(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)\big) \leqslant g\big(x_2, u^*(x_2)\big).$$

Hence, we have

$$-\big[b(x_2) + \partial_u f\big(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)\big)\big] \geqslant \delta. \tag{2.8}$$

From (2.7) and (2.8), we conclude that u^* is continuous.

We now show the uniqueness of nonzero solutions to (2.2) in X_+. Note that, since $f(x,0) = 0$, any such solution is bounded away from zero. We shall argue by contradiction. Let $\varphi_1 \neq \varphi_2$ be two nonzero solutions to (2.2) in X_+. Then it is easy to see that $k\varphi_1$ is a supersolution of (2.2) which is greater than both φ_1 and φ_2 provided k is sufficiently large. Actually k can be chosen so that $k\varphi_1 > \varphi_1 + \varphi_2$. Hence we may assume without loss of generality that $\varphi_2 \ll \varphi_1$. On the other hand, the fact that $g(x,\varphi_1) < g(x,\varphi_2)$ and Lemmas 2.2 and 2.4 yield

$$0 = \mathfrak{s}\big(L + g(x,\varphi_1)\big) < \mathfrak{s}\big(L + g(x,\varphi_2)\big) = 0.$$

This contradiction completes the proof. $\square$

Throughout the rest of this section, we shall focus on the case that $\Omega \subset \mathbb{R}$ and establish necessary and sufficient conditions for the existence of a steady state solution to (2.2). Besides the conditions (H1)–(H4), we assume that

(H5) $\Omega = (0, l)$ for some $0 < l < \infty$, and
(H6) $\partial_u f(x, 0)$ is Lipschitz continuous on $\overline{\Omega}$.

First, we need the following lemma which can be found in [14].

Lemma 2.7. *Assume that* (H1), (H2) *and* (H5) *hold and that $b \in X$ is Lipschitz continuous. Then L is a bounded, self-adjoint operator on H and has a simple eigenvalue λ_0 given by*

$$\lambda_0 = \max_{\|u\|_{L^2}=1} (Lu, u).$$

The maximum is attained by a strictly positive eigenfunction $\phi \in X$. Also $\sigma(L) \subset (-\infty, \lambda_0]$.

This gives another characterization of $\mathfrak{s}(L)$ and the next lemma gives a special condition for the nonexistence of positive solutions.

Lemma 2.8. *Suppose that* (H1)–(H6) *hold. In addition, suppose that b is Lipschitz continuous on $\overline{\Omega}$. If $\mathfrak{s}(L + g(x, 0)) \leqslant 0$, then* (2.2) *does not possess any solution in $X_+ \setminus \{0\}$.*

Proof. Suppose that w is a solution in $X_+ \setminus \{0\}$. By Lemma 2.2, we infer that $\mathfrak{s}(L + g(x, w)) = 0$. On the other hand, Lemmas 2.2 and 2.7 ensure that $\mathfrak{s}(L + g(x, 0))$ is a simple eigenvalue having an eigenfunction in $\operatorname{int} X_+$. By (H4) and Lemma 2.4, we have

$$0 = \mathfrak{s}\big(L + g(x, w)\big) < \mathfrak{s}\big(L + g(x, 0)\big) \leqslant 0.$$

The contradiction gives the desired conclusion. $\square$

Lemma 2.9. *Suppose that* (H1)–(H6) *hold and that $b \in X$ is Lipschitz continuous. The following statements are equivalent.*

(i) (2.2) *admits a subsolution $\underline{u} \in X_+ \setminus \{0\}$ which is not a solution, i.e., the expression on the left in* (2.4) *is not identically zero.*
(ii) (2.2) *admits an arbitrarily small subsolution in $X_+ \setminus \{0\}$ which is not a solution.*
(iii) $\mathfrak{s}(L + g(x, 0)) > 0$.

Proof. (i) $\Rightarrow$ (ii). This follows from the fact that

$$L\epsilon\underline{u} + f(x, \epsilon\underline{u}) \geqslant L\epsilon\underline{u} + \epsilon f(x, \underline{u}) \geqslant 0$$

for any $0 < \epsilon < 1$.

(ii) $\Rightarrow$ (i). This is trivial.

(i) $\Rightarrow$ (iii). Because of (i) and (H4),

$$\big(L + g(x, 0)\big)\underline{u} \geqslant L\underline{u} + g(x, \underline{u})\underline{u} = L\underline{u} + f(x, \underline{u}),$$

that is, $(L + g(x, 0))(-\underline{u}) \leqslant 0$ but not identically zero. As a consequence of Lemma 2.3, we have $\mathfrak{s}(L + g(x, 0)) \geqslant 0$. In fact, if $\mathfrak{s}(L + g(x, 0)) < 0$, by Lemma 2.3, $(L + g(x, 0))(-\underline{u}) \leqslant 0$

implies that $(-\underline{u}) \geqslant 0$, which is a contradiction. We next show $\mathfrak{s}(L+g(x,0)) \neq 0$. Suppose this is not the case. Let $\psi \in \operatorname{int} X_+$ be an eigenfunction corresponding to the eigenvalue 0, then

$$0=\big(\big(L+g(x,0)\big)\psi,-\underline{u}\big)=\big(\psi,\big(L+g(x,0)\big)(-\underline{u})\big)<0.$$

The contradiction shows $\mathfrak{s}(L+g(x,0))>0$.

(iii) $\Rightarrow$ (i). Again, let $\psi \in \operatorname{int} X_+$ be an eigenfunction associated with the eigenvalue $\mathfrak{s}(L+g(x,0))$, then $L\epsilon\psi+g(x,0)\epsilon\psi \gg 0$ for any $\epsilon>0$. By the continuity of $g(x,\cdot)$, we have

$$L\epsilon\psi+f(x,\epsilon\psi)=L\epsilon\psi+g(x,\epsilon\psi)\epsilon\psi>0$$

for sufficiently small $\epsilon>0$. Hence, $\epsilon\psi$ is a subsolution of (2.2). $\square$

Theorem 2.10. *Suppose that* (H1)–(H6) *hold and that $b \in X$ is Lipschitz continuous. If* (2.2) *has a supersolution $\tilde{u} \in X_+ \setminus \{0\}$, then* (2.2) *has a unique positive continuous solution if and only if* $\mathfrak{s}(L+g(x,0))>0$.

Proof. We first prove the necessity. Let $\varphi \in X_+ \setminus \{0\}$ be a positive solution to (2.2). Then we have, in fact, $\varphi \gg 0$ and $\mathfrak{s}(L+g(x,\varphi))=0$. Consequently,

$$\mathfrak{s}\big(L+g(x,0)\big)>\mathfrak{s}\big(L+g(x,\varphi)\big)=0.$$

Now, suppose that $\mathfrak{s}(L+g(x,0))>0$. Let ψ be the positive eigenfunction associated with $\mathfrak{s}(L+g(x,0))$, whose existence is guaranteed by the condition (H6) and Lemma 2.7. The proof of Lemma 2.9 shows that $\varepsilon\psi$ is a subsolution of (2.2) for sufficiently small ε. Also, $\tilde{u} \in X_+$ and $-(b(x)+g(x,\tilde{u}))\tilde{u} \geqslant J*\tilde{u} \gg 0$ force $\tilde{u} \gg 0$, and hence $\varepsilon\psi \ll \tilde{u}$ for some $\varepsilon>0$. It follows from Theorem 2.6 that (2.2) has a solution u in X_+ with $\varepsilon\psi \leqslant u \leqslant \tilde{u}$. $\square$

Corollary 2.11. *Assume* (H1)–(H6) *and that $b \in X$ is Lipschitz continuous. Suppose $\mathfrak{s}(L+g(x,0))>0$. Then the following statements are equivalent*:

(i) *Problem* (2.2) *has a solution in $X_+ \setminus \{0\}$.*
(ii) *Problem* (2.2) *has a supersolution in $X_+ \setminus \{0\}$.*
(iii) *Problem* (2.2) *has an arbitrarily large positive supersolution in X.*
(iv) *There exists $v \in \operatorname{int} X_+$, which is Lipschitz, such that $\mathfrak{s}(L+g(x,v)) \leqslant 0$.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of Theorem 2.10 and the fact that

$$L\alpha\omega+f(x,\alpha\omega)\leqslant\alpha\big(L\omega+f(x,\omega)\big)=0,$$

where $\alpha>1$ and $\omega \in X_+$ is a solution of (2.2). The equivalence of (ii) and (iii) comes from the fact that if ψ is a positive supersolution of (2.2), then

$$Lk\psi+f(x,k\psi)\leqslant Lk\psi+kf(x,\psi)\leqslant 0,\quad \text{for all } k>1.$$

If (2.2) has a positive solution w in X then $\mathfrak{s}(L+g(x,w))=0$. Let $v \in \operatorname{int} X_+$ be Lipschitz continuous. Due to (H3), $b(x)+g(x,v)$ is Lipschitz continuous. Lemmas 2.2 and 2.9 imply that $\mathfrak{s}(L+g(x,v))$ is an eigenvalue of the linear and bounded operator $L+g(x,v)I$ on X and there is a strictly positive eigenfunction associated with $\mathfrak{s}(L+g(x,v))$. Choose v such that $v \gg w$, by condition (H4), $g(x,\cdot)$ is non-increasing, it follows from Lemma 2.4 that

$$\mathfrak{s}\big(L+g(x,v)\big)<\mathfrak{s}\big(L+g(x,w)\big).$$

Therefore, (i) implies (iv). We now complete the proof by showing that (iv) implies (ii). Let ϕ be the strictly positive eigenfunction corresponding to $\mathfrak{s}(L + g(x, v))$. Then we have

$$Lk\phi + f(x, k\phi) = Lk\phi + kg(x, k\phi)\phi \leqslant Lk\phi + kg(x, v)\phi \leqslant 0$$

for sufficiently large k. Thus, $k\phi$ is the desired supersolution of (2.2). □

3. Existence and asymptotic behavior

In this section, we establish the basic existence and uniqueness results for (1.1) and study the long time behavior of the solution to (1.1). We shall first establish local existence and uniqueness in X.

For $t_1 > 0$, define $\widehat{X} = C([0, t_1], X)$ with norm $\|\phi\|_{\widehat{X}} = \max_{t\in[0,t_1]} \|\phi\|_X$.

Theorem 3.1. *Assume that* (H1)–(H3) *hold and $b \in X$. For each $\phi_0 \in X$, there exists $t_1 > 0$ such that* (1.1) *has a unique solution in $\widehat{X}$.*

Proof. We take the semigroup approach used in [5] to show the existence and uniqueness of solutions. As usual, we define the linear operator L on X by $L = J * u + b(x)u$. For each $\phi \in \widehat{X}$, we define mapping $S\phi = u$ where

$$u = e^{Lt}\phi(0) + \int_0^t e^{L(t-s)} f\bigl(x, \phi(s)\bigr)\, ds \tag{3.1}$$

and e^{Lt} is the semigroup on X generated by L, uniformly continuous because L is bounded on X. Now, Eq. (1.1) is reduced to (3.1). Since $f(x, \cdot)$ is locally Lipschitz, with an argument similar to that in [5], one can show that S is a contraction mapping on $\widehat{X}$ for the suitable t_1. Therefore, the existence and uniqueness of the solution to (1.1) follows from Banach's fixed point theorem. □

Definition 3.2. Let $S_T = \overline{\Omega} \times (0, T)$ for $0 < T \leqslant \infty$. A function $u \in C^1([0, \infty), X)$ is said to be a subsolution of (1.1) in S_T if

$$u_t \leqslant \int_\Omega J(x, y)u(y)\, dy + b(x)u(x) + f(x, u). \tag{3.2}$$

A supersolution is defined similarly by reversing the inequality.

Proposition 3.3. *Assume* (H1)–(H4) *are satisfied and that $b \in X$. Then* (1.1) *has a global solution $u(\cdot, x, \psi)$ for each $\psi \in X_+$.*

Proof. Let $\hat{u}$ be the solution to the linear equation

$$\begin{cases} u_t = \int_\Omega J(x, y)u(y)\, dy + b(x)u(x) + g(x, 0)u, \\ u(x, 0) = \psi, \end{cases}$$

where $g(x, 0)$ is given by (2.3). Since $L + g(x, 0)$ is a bounded linear operator on X, we have

$$\bigl\|\hat{u}(x, t, \psi)\bigr\|_X \leqslant \bigl\|e^{(L+g(x,0))t}\bigr\| \|\psi\|_X \leqslant e^{\|L+g(x,0)\|t} \|\psi\|_X.$$

This indicates that $\hat{u}$ is a global solution. Furthermore, $\hat{u} \in X_+$ according to the comparison principle, see [14]. Due to (H4) and (2.3)

$$f(x,u) = g(x,u)u \leqslant g(x,0)u,$$

whenever $u \geqslant 0$. Thus, $\hat{u}$ is a supersolution of (1.1). By the comparison principle,

$$0 \leqslant u(x,\cdot,\psi) \leqslant \hat{u}(x,\cdot,\psi),$$

where $u(x,\cdot,\psi)$ is the solution of (1.1) with $u(0)=\psi$. The desired conclusion follows immediately. □

Now we are ready to give the main result in this section.

Theorem 3.4. *Assume that* (H1)–(H6) *are satisfied and that $b \in X$ is Lipschitz continuous. Then one of following statements hold.*

(i) *If $\mathfrak{s}(L+g(x,0)) < 0$ then the zero solution of* (1.1) *is globally asymptotically stable in X_+.*
(ii) *If $\mathfrak{s}(L+g(x,0)) = 0$ then the solution $u(x,t,\psi)$ to* (1.1) *with $\psi \in X_+ \setminus \{0\}$ satisfies*

$$\lim_{t\to\infty} u(\cdot,t,\psi) = 0.$$

(iii) *If $\mathfrak{s}(L+g(x,0)) > 0$ then* (1.1) *admits at most one stationary solution in* $\operatorname{int} X_+$. *If* (1.1) *has a stationary solution $\tilde{u} \in \operatorname{int} X_+$ then $\tilde{u}$ is globally asymptotically stable in X_+.*
(iv) *If $\mathfrak{s}(L+g(x,0)) > 0$ and* (1.1) *has no positive stationary solution then*

$$\lim_{t\to\infty} \|u(\cdot,t,\phi_0)\|_X = \infty \quad \textit{for all } \phi_0 \in X_+ \setminus \{0\}.$$

Proof. (i) By (H4) and the comparison principle given in [14], $0 \leqslant u(x,t,\phi) \leqslant e^{(L+g(x,0))t}\phi$, where $u(x,t,\phi)$ is the solution to (1.1) with initial data $\phi \in X_+ \setminus \{0\}$ (see proof of Proposition 3.3). Since $\mathfrak{s}(L+g(x,0)) < 0$, for some $M, \alpha > 0$

$$\|e^{t(L+g(x,0))}\|_X \leqslant Me^{-\alpha t}$$

(see [11] or [17]). Thus, we find $\lim_{t\to\infty} \|u(\cdot,t,\phi)\|_X = 0$.

(ii) Let φ be the eigenfunction associated with $\mathfrak{s}(L+g(x,0))$. Because of $\mathfrak{s}(L+g(x,0)) = 0$, $k\varphi$ is a supersolution of (1.1) if $k > 0$. By the comparison principle, we have

$$u(\cdot,t+h,k\varphi) = u\big(\cdot,t,u(\cdot,h,k\varphi)\big) \leqslant u(\cdot,t,k\varphi)$$

for each $h > 0$, that is, the function $t \mapsto u(\cdot,t,k\varphi)$ is non-increasing. This ensures that the pointwise limit

$$\hat{u} = \lim_{t\to\infty} u(\cdot,t,k\varphi) \tag{3.3}$$

exists. By the monotone convergence theorem and the continuity of f, $L\hat{u} + f(x,\hat{u}) = 0$ and $0 \leqslant \hat{u} \leqslant k\varphi$. It follows from Lemma 2.8 that $\hat{u} = 0$. Let $\psi \in X_+ \setminus \{0\}$, then there exists $k > 0$ such that $0 \leqslant \psi \ll k\varphi$. Again, the comparison principle gives $0 \leqslant u(\cdot,t,\psi) \leqslant u(\cdot,t,k\varphi)$, for all $t \geqslant 0$. (ii) follows from this fact together with (3.3).

(iii) Suppose that (2.2) has a solution $\tilde{u} \in X_+ \setminus \{0\}$. By Lemma 2.9 and Corollary 2.11, Eq. (1.1) has an arbitrarily small subsolution $\epsilon\varphi$ and an arbitrarily large supersolution $k\tilde{u}$, where φ is the strictly positive eigenfunction corresponding to $\mathfrak{s}(L+g(x,0))$ and $0 < \epsilon < 1 < k$.

With reasoning similar to that for (ii), we find that the function $u(x,\cdot,\epsilon\varphi)$ is nondecreasing while $u(x,\cdot,k\tilde{u})$ is non-increasing. Furthermore, the comparison principle implies that $k\tilde{u} \geqslant u(\cdot,t,k\tilde{u}) \geqslant u(\cdot,t,\epsilon\varphi) \geqslant \epsilon\varphi$ for all $t \geqslant 0$. The uniqueness of the positive stationary solution of (1.1) together with our previous argument yields that the pointwise convergence

$$\lim_{t\to\infty} u(\cdot,t,\epsilon\varphi) = \tilde{u}, \qquad \lim_{t\to\infty} u(\cdot,t,k\tilde{u}) = \tilde{u},$$

both hold true. Because $\tilde{u}$ is continuous, Dini's theorem gives

$$\lim_{t\to\infty} \left\| u(\cdot,t,\epsilon\varphi) - \tilde{u} \right\|_X = 0, \qquad \lim_{t\to\infty} \left\| u(\cdot,t,k\tilde{u}) - \tilde{u} \right\|_X = 0 \tag{3.4}$$

(see [7]). For $\phi \in X_+ \setminus \{0\}$, by the comparison principle in [14], there exists $h^* > 0$ such that $u(\cdot,h^*,\phi) \gg 0$. Moreover, we have that $u(\cdot,t,\underline{\gamma}\varphi) \leqslant u(\cdot,t+h^*,\phi)$ for some $0 < \underline{\gamma} < 1$ and $u(\cdot,t,\phi) \leqslant u(\cdot,t,\overline{\gamma}\tilde{u})$ for some $\overline{\gamma} > 1$. Therefore, (iii) follows from (3.4).

(iv) Corollary 2.11 implies

$$\mathfrak{s}\big(L + g(x,v)\big) > 0$$

for each $v \in \operatorname{int} X_+$ which is Lipschitz. It is well known that $L + g(x,v)I$ generates a uniformly continuous semigroup $e^{t(L+g(x,v)I)}$ on X. Let $\psi^* \in \operatorname{int} X_+$ be an eigenfunction corresponding to $\mathfrak{s}(L+g(x,v))$. By the spectral mapping theorem, $\sigma(e^{t(L+g(x,v))}) = e^{t\sigma(L+g(x,v)I)}$ and $e^{t(L+g(x,v)I)}\psi^* = e^{t\mathfrak{s}(L+g(x,v))}\psi^*$. Let $\phi_0 \in \operatorname{int} X_+$ and choose $\rho > 0$ such that $\rho\psi^* \leqslant \phi_0$. The comparison principle gives

$$e^{t(L+g(x,v)I)}\phi_0 \geqslant e^{t(L+g(x,v)I)}\rho\psi^* = \rho e^{t\mathfrak{s}(L+g(x,v))}\psi^*. \tag{3.5}$$

The above inequality implies

$$\left\| e^{t(L+g(x,v)I)}\phi_0 \right\|_X \geqslant \left\| \rho e^{t\mathfrak{s}(L+g(x,v)I)}\psi^* \right\|_X = \rho e^{t\mathfrak{s}(L+g(x,v)I)} \|\psi^*\|_X$$

and since $\mathfrak{s}(L + g(x,v)I) > 0$,

$$\lim_{t\to\infty} \left\| e^{t(L+g(x,v)I)}\phi_0 \right\|_X = \infty \tag{3.6}$$

for any $\phi_0 \in \operatorname{int} X_+$. Note that (3.6) also holds for $e^{t(L+g(x,v)I)}\varphi_0$ with $\varphi_0 \in X_+ \setminus \{0\}$ because $e^{h(L+g(x,v)I)}\varphi_0 \gg 0$ for some $h > 0$ (see [14]).

In the following, we shall adopt an idea from [8] to complete the proof. We first prove that

$$\lim_{t\to\infty} \left\| u(\cdot,t,\epsilon\varphi) \right\|_X = \infty,$$

where φ is the strictly positive eigenfunction corresponding to $\mathfrak{s}(L + g(x,0))$ and $0 < \epsilon$ is small.

Suppose this is not true. Thanks to the monotonicity of $u(x,\cdot,\epsilon\varphi)$ (see the proof of part (iii), above), there exists an constant $c > 0$ such that

$$\left\| u(\cdot,t,\epsilon\varphi) \right\|_X \leqslant c < \infty$$

for all $t \geqslant 0$. Consequently,

$$\begin{aligned} u_t(\cdot,\cdot,\epsilon\varphi) &= Lu(\cdot,\cdot,\epsilon\varphi) + f\big(x, u(\cdot,\cdot,\epsilon\varphi)\big) \\ &= Lu(\cdot,\cdot,\epsilon\varphi) + g\big(x, u(\cdot,\cdot,\epsilon\varphi)\big)u(\cdot,\cdot,\epsilon\varphi) \\ &\geqslant Lu(\cdot,\cdot,\epsilon\varphi) + g(x,c)u(\cdot,\cdot,\epsilon\varphi) \end{aligned}$$

in $\overline{\Omega} \times (0,\infty)$. This fact and the comparison principle imply that

$$e^{t(L+g(x,c))}\epsilon\varphi \leqslant u(\cdot,t,\epsilon\varphi)$$

for $t \geqslant 0$. Thus, (3.6) yields

$$\lim_{t\to\infty} \left\| u(\cdot, t, \epsilon\varphi) \right\|_X = \infty.$$

The contradiction gives the desired the conclusion. For any $w_0 \in X_+ \setminus \{0\}$, as mentioned in the proof of part (iii), there exists $h^* > 0$ such that $u(\cdot, t, \epsilon\varphi) \leqslant u(\cdot, t + h^*, w_0)$ for some $0 < \epsilon < 1$. Thus, the proof is complete. □

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